METHOD OF POINCARÉ IN THE THEORY OF NONLINEAR OSCILLATIONS CASES OF DEGENERATION OF THE SYSTEM

PMM Vol. 31, No. 4, 1967, pp. 615-623

A.P. PROSKURIAKOV
(Moscow)

(*Received December 13, 1966*)

As we know, the purpose of Poincaré's method is to construct periodic solutions of nonlinear systems, differential equations of which contain a small parameter. We assume that when this small parameter becomes equal to zero, the order of the system does not change.

The method was first proposed by Poincaré in the early 90's of the last century, as an aid in solving problems in celestial mechanics [1 and 2]. Later, the method found applications in other fields such as general mechanics, electrical engineering and physics.

At present, Poincaré's method remains one of the basic tools for investigating nonlinear oscillations. Initially it was used mainly in conjunction with quasi-linear systems, but gradually it had spread to other forms of nonlinear systems while continuing to undergo various improvements.

The method is based on a special manner of selecting initial conditions for the system in question. The choice is subject to the requirement that conditions of periodicity of solutions are fulfilled. From now on, we shall use the name "Poincaré's method" to describe a method of constructing periodic solutions, which is based on the original Poincaré's idea of selection of initial conditions.

A large amount of literature on the Poincaré's method exists. We shall only mention books of Andronov, Vitt and Khaikin [3], Bulgakov [4] and Malkin [5 and 6], all on nonlinear oscillations. A book by Duboshin [7] on celestial mechanics contains an exposition of the method which is closest to its original form. From foreign literature we mention Minorsky [8].

It should be noted that no attention was paid, during the development of the method, to cases which eventually necessitated the system to be treated as degenerate.(*) These were found, however, to be cases of apparent degeneracy which could be removed. Proofs of fundamental theorems of existence of solutions were found, in these cases, to be much more complex and construction of periodic solutions also encountered difficulties.

In this paper we adopt a somewhat different method of application of the basic idea of Poincaré. We take into account all conditions leading to the possibility of fictitious degeneracy of the system. Next, by avoiding these cases we are able to simplify the method considerably. Later when we derive the method, we shall refer to several works illustrating applications of the simplified method to various problems.

First we shall consider a nonautonomous nonlinear system containing n first order Eqs.

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n, \mu) \quad (s = 1, \ldots, n) \tag{1}$$

Let X_s be analytic functions of $x_1, \ldots, x_n$ and μ sin some region of variation of $x_1, \ldots, x_n$

*) We shall use the name "degenerate system" to describe a system in which the functional determinant of auxilliary equations obtained from the conditions of periodicity and referring to increments in initial values of the generating system, becomes equal to zero. See Formula (18) below.

when $0 \leq \mu < \mu_0$ and let X_s be, in addition, continuous periodic functions of time with period equal to 2π.

When $\mu = 0$, system (1) becomes a so called generating system. Let a general solution of the generating system be $x_{s0}(t)$ and let us denote initial values of the generating solution by

$$x_{s0}(0) = h_s \qquad (s = 1, \ldots, n) \tag{2}$$

When any one of the equations of the generating system has a term on the right-hand side which is a function of t only, then initial conditions can be given in a slightly different form (see e.g. [9])

$$x_{s0}(0) = a_s + h_s \tag{3}$$

where a_s are known and h_s are parameters to be determined. These parameters will not, in general, coincide with arbitrary constants of integration, but the amount of parameters equals the amount of constants. Therefore we can find the constants of integration if we know h_s.

Two cases are possible. In the first case a general solution of the generating system is periodic, and such a solution represents a family of solutions dependent on n parameters h_s. In the second case the solution is not periodic, whereupon we assume that either one or several families of periodic solutions dependent on m $(m < n)$ parameters h_s or, one or several isolated periodic solutions can be separated from the general nonperiodic solution.

Clearly, the purpose of Poincaré's method is to construct all periodic solutions of (1) which, at $\mu = 0$, become periodic solutions of the generating system.

From this it follows that when periodic solutions are singled out from the general solution of the generating system, it is important that they should embrace all possible periodic solutions possessing the same period.

Thus some of the parameters or, in the case of isolated solutions, all parameters h_s, can be found from the conditions of periodicity of the generating solution.

In the general case initial conditions for (1) are written as

$$x_s(0) = a_s + h_s + \beta_s \tag{4}$$

where β_s are small additional terms dependent on μ and becoming zero when $\mu = 0$. In accordance with Poincaré's idea, choice of h_s and β_s implements the construction of periodic solutions of the system.

It should be pointed out that parameters h_s and β_s always enter initial conditions as a sum $h_s + \beta_s$. We have said before that some of the parameters h_s or, in the case of isolated solutions of the generating system, all of them can be determined in advance. However, in what follows, we shall regard all h_s as unknowns.

The above mentioned property of parameters h_s and β_s was not used before(*). Poincaré and other authors who elaborated or just applied Poincaré's method all used h_s as unconnected with β_s.

We note that in some problems initial conditions are independent only for first k functions $x_s(t)$. For the remaining n-k functions, initial conditions depend on initial conditions of the preceding functions and on the parameter μ, i.e.

$$x_s(0) = a_s + \varphi_{s-k}(h_1 + \beta_1, \ldots, h_k + \beta_k, \mu) \qquad (s = k+1, \ldots, n) \tag{5}$$

Here φ_{s-k} are analytic functions of their arguments, vanishing when $\mu = 0$. The amount of k may range, in various problems, from 1 to n. Such a case was first investigated by Malkin [5 and 6] when constructing periodic solutions of quasi-linear nonautonomous systems described by first order equations when not all natural frequencies of the generating system are equal either to zero or to an integer. The analogous case with second order equations was investigated in [12]. Such a case is also encountered in autonomous quasi-linear systems when not all natural frequencies are mutually commeasurable [13].

The above property of h_s and β_s has important corollaries. Since (1) is analytic, its solution can be given in terms of analytic functions of independent initial conditions, and of a small parameter μ. We shall write this solution as

---

*) This property was utilized first in the author's papers on quasi-linear systems and then in Kopnins papers on almost arbitrary systems [10 and 11].

$$x_s(t) = \sum_{m=0}^{\infty} C_{sm}(t, h_1 + \beta_1, \ldots, h_k + \beta_k)\mu^m \qquad (s = 1, \ldots, n) \tag{6}$$

When $\beta_1 = \ldots = \beta_k = 0$, functions $C_{s0}(t)$ become $x_{s0}(t)$. Moreover when $t = 0$, we have

$$C_{s0}(0) = a_s + h_s + \beta_s, \qquad C_{sm}(0) = 0 \qquad (m = 1, 2, \ldots) \tag{7}$$

Conditions of periodicity of solution of (1) can be written as

$$\psi_s(h_1 + \beta_1, \ldots, h_k + \beta_k, \mu) = x_s(2\pi) - x_s(0) = 0 \qquad (s = 1, \ldots, n) \tag{8}$$

Functions ψ_s are analytic functions of their arguments. First k conditions of periodicity yield h_s and β_s. Remaining n-k conditions are used to determine initial functions φ_{s-k} when $s = k + 1, \ldots, n$. Thus, construction of periodic solutions of (1) by Poincaré's method is reduced to determination of k parameters h_s, and k functions $\beta_s(\mu)$ from Eqs.

$$\psi_s(h_1 + \beta_1, \ldots, h_k + \beta_k, \mu) = 0 \qquad (s = 1, \ldots, k) \tag{9}$$

Some of the functions ψ_s may have the form $\psi_s = \mu\psi_s{}^*$, i.e. may contain μ as a multiplier. Corresponding conditions of periodicity are then separated into

$$\mu = 0, \qquad \psi_s{}^* = 0$$

The first of them means that function $x_{s0}(t)$ is periodic for all values of h_s, while the second condition can be written as

$$\psi_s{}^* = \sum_{m=1}^{\infty} C_{sm}(2\pi, h_1 + \beta_1, \ldots, h_k + \beta_k)\mu^{m-1} = 0 \tag{10}$$

Let us divide (when it is possible) the relations representing initial periodicity conditions by μ, and neglect the superscript *, while retaining the previous notation for ψ_s.

Since functions $\beta_s(\mu)$ have the property that $\beta_s(0) = 0$, h_s should satisfy the following Eqs.:

$$\psi_s(h_1, \ldots, h_k, 0) = 0 \qquad (s = 1, \ldots, k) \tag{11}$$

Let us consider the left-hand sides of these equations in more detail. Two cases are possible. In the first case the corresponding function $x_{s0}(t)$ is periodic at any h_s. Then,

$$\psi_s(h_1, \ldots, h_k, 0) = C_{s1}(2\pi, h_1, \ldots, h_k) \tag{12}$$

In the second case $x_{s0}(t)$ is not periodic [14] and consequently

$$\psi_s(h_1, \ldots, h_k, 0) = x_{s0}(2\pi, h_1, \ldots, h_k) - (a_s + h_s) \tag{13}$$

Some of Eqs. of (11) may become identities [6 and 15]. This is possible only in the first case when the left-hand side of equation is defined by (12). But in this case the magnitudes $C_{s1}(2\pi, h_1 + \beta_1, \ldots, h_k + \beta_k)$ will also be identically equal to zero. Let us divide the corresponding Eqs. (9) by μ once more and again insert $\beta_1 = \ldots = \beta_k = \mu = 0$ into them. We shall then have

$$\psi_s(h_1, \ldots, h_k, 0) = C_{s2}(2\pi, h_1, \ldots, h_k) \tag{14}$$

We shall use these equations to replace the identities. Should any equation of the type (14) again become an identity, we must repeat the above transformation once more, etc. If we come to the conclusion that the available periodicity condition cannot yield an equation for parameters $h_1, \ldots, h_k$, then one of the sums $h_s + \beta_s$ must remain undetermined. This may arise e.g. when the system (1) has an analytic first integral [2 and 16].

There are cases when all the roots cannot be obtained from the given system of Eqs. (11). Then, some equations must be replaced with other equations obtained from combinations of conditions of periodicity. We met such a case when an example of a quasi-linear nonautonomous system with one degree of freedom was considered [17].

Eqs. (11) enable us to find parameters h_s, the latter being initial amplitudes of the generating solution or of a certain part of it which corresponds to its natural oscillations in case of quasi-linear systems. We shall call these equations the amplitude equations. Assume now that amplitude equations possess one or several solutions, some of which may be repeated

$$h_1 = h_1{}^*, \ldots, h_k = h_k{}^* \tag{15}$$

Let us insert one of these solutions into (9). We obtain

$$\psi_s(h_1{}^* + \beta_1, \ldots, h_k{}^* + \beta_k, \mu) = 0 \qquad (s = 1, \ldots, k) \tag{16}$$

The left-hand sides of these equations become zero when $\beta_1 = \ldots = \beta_k = \mu = 0$. Thus the problem is reduced to the following problem of the theory of implicit functions: to obtain from (16) the magnitudes β_s in terms of μ under the condition that $\beta_s(0) = 0$.

We shall now construct a functional determinant of this system. When $\mu \neq 0$, we have an obvious relation

$$\partial\psi_s / \partial\beta_r = \partial\psi_s / \partial h_r \qquad (s, r = 1, \ldots, k) \tag{17}$$

When $\beta_1 = \ldots = \beta_k = \mu = 0$, derivatives $\partial\psi_s/\partial h_r$ should be computed for $h_r = h_r^*$, therefore the required functional determinant is

$$\Delta = \begin{vmatrix} \partial\psi_1/\partial h_1 \ldots \partial\psi_1/\partial h_k \\ \cdots\cdots\cdots\cdots \\ \partial\psi_k/\partial h_1 \ldots \partial\psi_k/\partial h_k \end{vmatrix}_{\beta_r=\mu=0,\ h_r=h_r^*} \tag{18}$$

If, for any solution of (11), this determinant is different from zero, then the solution is simple. We know that in this case there exists a corresponding unique analytic solution of (16)

$$\beta_s = \sum_{m=1}^{\infty} b_{sm}\mu^m \tag{19}$$

which satisfies the conditions $\beta_s(0) = 0$.

If $\Delta = 0$, then a corresponding solution of (11) is repeated. If the matrix of elements included in (18) has a rank $(k\text{-}1)$, then $(k\text{-}1)$ magnitudes $h_s^* + \beta_s$ can be eliminated from (16), yielding

$$\Phi\,(h_r^* + \beta_r,\ \mu) = 0 \tag{20}$$

where the subscript r falls within the range $1, \ldots, k$ and the problem is reduced to determination of β_r as an implicit function of a small parameter μ.

The solution of this problem is based on the Quisé theory of algebraic functions and on the Weierstrass' theorem. Different branches $\beta_r(\mu)$ are given by series in whole or fractional powers of μ. The number of these branches and a possible form of their expansion in μ are determined by the multiplicity of solutions of amplitude equations. Multiplicity of solutions is, in turn, connected with the order of the first derivative $\partial^n\Phi/\partial\beta_r^n$ which is not vanishing. Having determined β_r, we can easily find the remaining parameters.

This method is practical if the number of Eqs. in (16) is small. An example illustrating its use is given in [9], where the method is applied to a quasi-linear nonautonomous system with one degree of freedom, the system consisting of two equations. A detailed analysis of Eq. (20) itself is given for the case of double and triple roots of the amplitude equation of a quasi-linear autonomous system [18].

The general method of determination of β_s as implicit functions of μ, in cases when the rank of the matrix mentioned previously is lower than $(k\text{-}1)$ or, when the rank is equal to $(k\text{-}1)$ but the number of equations is large, was investigated by Mac-Millan in [19 and 20].

In order to determine the form assumed by the power series into which β_s are expanded and to establish the degree in which first terms of these series appear, he transformed the system (16) of transcendental equations, reducing it to an equivalent system of algebraic equations, the left-hand sides of which were finite sums of homogeneous polynomials in β_s, and he applied the Newton's quadrature method to this system.

When all elements of the functional determinant vanish, special methods can be used to determine β_s [14].

When expansions of all β_s into series in integral or fractional powers of μ are known, we can easily obtain final expansions of functions $x_s(t)$ into series in μ. Obviously, their form depends on the form of expansions of $\beta_s(\mu)$. It is possible to obtain general formulas for some of the first coefficients of such series, or else they can be determined by consecutive integration of differential equations set up for these coefficients.

Poincaré [2] mentioned some particular cases. They are: 1) when the system (1) has, in addition to periodic solutions of period 2π, other solutions of period $2\pi m$ where m is an integer (subharmonic oscillations) and 2) when the system (1) has one or several analytic first integrals. In the latter case, periodic solutions of the system will depend on one or several arbitrary magnitudes $h_s + \beta_s$ corresponding to the number of first integrals.

Let us now consider an autonomous nonlinear system

$$\frac{dx_s}{dt} = X_s(x_1, \ldots, x_n, \mu) \qquad (s = 1, \ldots, n) \tag{21}$$

We shall retain our assumptions referring to the functions X_s as they are for nonautonomous system. Initial conditions of the generating system will be given by Formula (3).

Poincaré's method will, in this case, exhibit some specific features resulting from the peculiar properties of autonomous systems. The first peculiar property is that an autonomous system retains its form when time t is replaced with $t + h_0$ where h_0 is arbitrary. Consequently, the solution of (21) depends on h_0, which can be found from one of the initial conditions. Therefore any of the sums $h_s + \beta_s$ can be chosen and, assuming that the number of independent initial conditions is equal to k, just as in the nonautonomous case, we can put e.g.,

$$h_k = 0, \qquad \beta_k = 0 \tag{22}$$

The second feature of the method is that period T of the solution of an autonomous system is not a specified value and it depends on μ and initial conditions. Consequently, periodicity conditions for autonomous systems will be

$$\psi_s(T, h_1 + \beta_1, \ldots, h_{k-1} + \beta_{k-1}, \mu) = x_s(T) - x_s(0) = 0 \qquad (s = 1, \ldots, n) \tag{23}$$

where ψ_s are analytic functions of their arguments.

First k conditions give $(k\text{-}1)$ sums $h_s + \beta_s$ and the period T, while the remaining $(n\text{-}k)$ conditions give initial functions φ_{s-k}.

When $\beta_1 = \ldots = \beta_{k-1} = \mu = 0$, we have k equations defining the period T_0 of the generating solution and parameters $h_1, \ldots, h_{k-1}$.

$$\psi_s(T_0, h_1, \ldots, h_{k-1}, 0) = 0 \qquad (s = 1, \ldots, k) \tag{24}$$

In order to bring the problem to the form which was obtained in the nonautonomous case, we shall eliminate the period T from the first k conditions of periodicity (23). Let us calculate the derivative of ψ_s with respect to T when $\mu = 0$, Taking into account the property of last $(n\text{-}k)$ initial conditions and Eq. (22), we obtain

$$(\partial\psi_s / \partial T)_0 = \dot{x}_{s0}(T_0) = \dot{x}_{s0}(0) = X_{s0}^*(h_1, \ldots, h_{k-1})$$

Functions X_{s0}^* cannot become zero, since in this case parameters $h_1, \ldots, h_{k-1}$ would be connected by additional relations contradicting (24). Consequently, period T can always be found from first k conditions of periodicity in the form of an analytic function of independent initial conditions and parameter μ. The k-th condition of periodicity yields

$$T = T(h_1 + \beta_1, \ldots, h_{k-1} + \beta_{k-1}, \mu) \tag{25}$$

Substituting T into the remaining $(k\text{-}1)$ conditions, we obtain

$$\Psi_s(h_1 + \beta_1, \ldots, h_{k-1} + \beta_{k-1}, \mu) = 0 \qquad (s = 1, \ldots, k-1) \tag{26}$$

where Ψ_s are also analytic functions of their arguments.

Further steps of the solution are the same as far nonautonomous systems. The number of equations and unknowns will however be smaller by one and derivatives of Ψ_s, with respect to h_r and β_r, will have to be calculated with Ψ_s taken as function of a function [10], e.g.

$$\frac{\partial \Psi_s}{\partial h_r} = \frac{\partial \psi_s}{\partial h_r} + \frac{\partial \psi_s}{\partial T}\frac{\partial T}{\partial h_r} \qquad (s, r = 1, \ldots, k-1) \tag{27}$$

Period of an autonomous system can be written as a sum

$$T = T_0 + \alpha \tag{28}$$

where $T_0 = T_0(h_1, \ldots, h_{k-1})$ is the period of the generating solution and α becomes zero when $\mu = 0$. When the system is quasi-linear, T_0 is independent of initial conditions. A method for elimination of α is given for such systems in e.g. [18].

When analytic first integrals which are time independent [2] appear in the autonomous system, periodic solutions will depend, as in the nonautonomous case, on the corresponding number of arbitrary magnitudes $h_s + \beta_s$. The existence of even one first integral makes the construction of a periodic solution with a given period possible.

When the solution is expanded into a series in integral or fractional powers of μ, we must replace time t with another variable beforehand, since the period T in time t depends on μ. A change of variable is performed so as to make the period independent of μ and equal to the period of the generating solution. This enables us to represent the solution in the form of a series in integral or fractional powers of μ, whose coefficients have a period independent of μ [5 and 18].

Summing it all up we must note that fictitious degeneration of the system (determinant Δ becoming equal to zero) may be due to the following causes: 1) failure to take into consideration the link between the parameters h_s and β_s; 2) existence of dependence between initial conditions in different equations; 3) appearance of μ as a multiplier in any one condition of periodicity; 4) existence of special cases of amplitude equations. If all these causes are removed, then the system degenerates, in fact, in only two cases: when amplitude equations have repeated solutions, or when the system has first integrals.

We can also interpret Poincaré's method as one establishing the extension of periodic solutions of the generating system to the initial system. We have shown previously that solutions of the generating system can either be isolated or else they may form a family of solutions depending on some parameters. However, if the initial system has no first integrals and if amplitude equations have a finite number of solutions, then only a finite number of periodic solutions of the generating system can become periodic solutions of the initial system. On the other hand, some periodic solutions of the generating system may have more than one corresponding solution of the initial system. The principal aim of Poincaré's method is to determine these calues of h_s, which establish one-to-one correspondence between these periodic solutions of the generating and initial systems, which pass from one into the other. Multiple solutions of amplitude equations are all counted as separate, and their number defines the degree of multiplicity.

The problem of convergence of series representing periodic solutions has not, so far, received sufficient attention. Proofs for theorems on the convergence of these series for nonautonomous systems of general type when μ is sufficiently small, are given in [6]. Estimates of radii of convergence of series or estimates of degrees of approximation achieved by use of the first few terms, are given only fo a few isolated cases and are, as a rule, not very effective. Nevertheless, in spite of the fact that only a few approximations are obtained in practice and that no effective method exists for obtaining estimates, Poincaré's method i is completely rigorous.

In conclusion we note that modifications of Poincaré's method exist. Here we may mention tne method of selecting arbitrary constants of integration due to Bulgakov [4]. Another unquestionable modification due to Shimanov [21] results in the method of auxilliary functions, which is useful in proofs of existence of periodic solutions in various cases.

BIBLIOGRAPHY

1. Poincaré, H., Sur le problème de trois corps et les équations de la dynamiques. Acta Mathematica, t.13, 1890.
2. Poincaré, H., Les Méthodes Nouvelles de la Mécanique Céleste. T. I, Paris, 1892.
3. Andronov, A.A., Vitt, A.A. and Khaikin, S.E., Theory of Oscillations, Fizmatgiz, 1959.
4. Bulgakov, B.V., Oscillations, GITTL, 1954.
5. Malkin, I.G., Methods of Liapunov and Poincaré in the Theory of Nonlinear Oscillations, GIITL, 1949.
6. Malkin, I.G., Certain problems of Nonlinear Oscillation Theory GITTL, 1956.
7. Duboshin, G.N., Celestial Mechanics. Analytical and Qualitative Methods, Izd. "Nauka", 1964.
8. Minorsky, N., Nonlinear Oscillations. New York, 1962.
9. Proskuriakov, A.P., On constructing periodic solutions of quasi-linear non-self-contained systems with one degree of freedom, when amplitude equations possess multiple roots, PMM, Vol. 30, No. 6, 1966.
10. Kopnin, Iu.M., Oscillations of autonomous systems with one degree of freedom, Inzh. zh. Vol. 2, No. 3, 1962.
11. Kopnin, Iu.M., Periodic oscillations of nonlinear nonautonomous systems with many degrees of freedom, Inzh. zh., Vol. 5, No. 2, 1965.
12. Plotnikova, G.V. and Bainov, D.D., Periodic oscillations of a mechanical system with n degrees of freedom, in the presence of resonant frequencies, Rev. Roumaine des

Sci. Techn. Mechanique Appl., Vol. 10, No. 2, 1965.
13. Proskuriakov, A.P., On the construction of periodic solutions of quasi-linear autonomous systems with several degrees of freedom, PMM Vol. 26, No. 2, 1962.
14. Proskuriakov, A.P., Periodic solutions of quasi-linear non-self-contained systems with one degree of freedom in particular cases, PMM Vol. 31, No. 2, 1967.
15. Merman, G.A., A new class of periodic solutions in the limited three-body problem and in the Hill's problem, Trudy Inst. teoret. astronomii, No. 1, 1952.
16. Arkhangel'skii, Iu.A., Periodic solutions of quasi-linear autonomous systems which have first integrals, PMM, Vol. 27, No. 2, 1963.
17. Arkhangel'skii, Iu.A., On a certain case of construction of periodic solutions of quasi-linear systems, PMM, Vol. 28, No. 5, 1964.
18. Proskuriakov, A.P., Periodic solutions of quasi-linear autonomous systems with one degree of freedom in the form of infinite series with fractional powers of the parameter, PMM, Vol. 25, No. 5, 1961.
19. Mac-Millan, W.D., A reduction of a system of power series to an equivalent system of polynomials. Mathematische Annalen, Bd. 72, 1912.
20. Mac-Millan, W.D., A method for determining the solution of a system of analytic function in the neighborhood of a branch point. Mathematische Annalen, Bd. 72, 1912.
21. Shimanov, S.N., Oscillations of quasi-linear systems with nonanalytic nonlinearity characteristic, PMM, Vol. 21, No. 2, 1957.

Translated by L.K.